<u>EJERCICIO 1</u> (minuto 13:10)

Demostrar que:

$$\left\{\boldsymbol{\phi}_{(t,\vec{x})}\,;\boldsymbol{\phi}_{(t,\vec{y})}\right\}=\mathbf{0}$$

$$\left\{\boldsymbol{\pi}_{(t,\vec{x})}\,;\boldsymbol{\pi}_{(t,\vec{y})}\right\}=\mathbf{0}$$

Fórmula 38.6 del Formulario de Crul del Curso de Mecánica Teórica de Javier

$$\{A\,;B\}=\int dx\left(\frac{\delta A}{\delta\phi_{(x)}}\frac{\delta B}{\delta\pi_{(x)}}-\frac{\delta A}{\delta\pi_{(x)}}\frac{\delta B}{\delta\phi_{(x)}}\right)$$

Haciendo un análisis unidimensional, para un dado tiempo t

$$\left\{\phi_{(x)}\,;\phi_{(y)}\right\}=\int dz\left(\frac{\delta\phi_{(x)}}{\delta\phi_{(z)}}\frac{\delta\phi_{(y)}}{\delta\pi_{(z)}}-\frac{\delta\phi_{(x)}}{\delta\pi_{(z)}}\frac{\delta\phi_{(y)}}{\delta\phi_{(z)}}\right)$$

Si tenemos un funcional F tal que depende de ϕ, y que da como imagen el valor del campo en x:

$$F_{[\phi_{(x)}]}=\phi_{(x)}$$

Calcular la derivada respecto a un campo nuevo, π, del que el funcional no depende, en otro punto sería:

$$\frac{\delta\phi_{(x)}}{\delta\pi_{(z)}}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\phi_{(x)}\big|_{\pi+\varepsilon}-\phi_{(x)}\big|_{\pi}\right)$$

Pero como decíamos, $\phi_{(x)}$ no depende de π, por lo que el límite tiende a 0

$$\frac{\delta\phi_{(x)}}{\delta\pi_{(z)}}=0$$

$$\left\{\phi_{(x)}\,;\phi_{(y)}\right\}=\int dz\left(\frac{\delta\phi_{(x)}}{\delta\phi_{(z)}}0-0\frac{\delta\phi_{(y)}}{\delta\phi_{(z)}}\right)$$

$$\boxed{\left\{\boldsymbol{\phi}_{(x)}\,;\boldsymbol{\phi}_{(y)}\right\}=\mathbf{0}}$$

Del mismo modo:

$$\left\{\pi_{(x)}\,;\pi_{(y)}\right\}=\int dz\left(\frac{\delta\pi_{(x)}}{\delta\phi_{(z)}}\frac{\delta\pi_{(y)}}{\delta\pi_{(z)}}-\frac{\delta\pi_{(x)}}{\delta\pi_{(z)}}\frac{\delta\pi_{(y)}}{\delta\phi_{(z)}}\right)$$

$$\left\{\pi_{(x)}\,;\pi_{(y)}\right\}=\int dz\left(0\frac{\delta\pi_{(y)}}{\delta\pi_{(z)}}-\frac{\delta\pi_{(x)}}{\delta\pi_{(z)}}0\right)$$

$$\boxed{\left\{\boldsymbol{\pi}_{(x)}\,;\boldsymbol{\pi}_{(y)}\right\}=\mathbf{0}}$$

<u>EJERCICIO 2</u> (21:49)

Calcular:

$$\left[\boldsymbol{\phi}_{(t_1,\vec{x}_1)}\ ;\boldsymbol{\phi}_{(t_2,\vec{x}_2)}\right]$$

$$\phi_{(t_1,\vec{x}_1)}=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right)$$

$$\phi_{(t_2,\vec{x}_2)}=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_{(\vec{k})}e^{-ikx_2}+a^{\dagger}_{(\vec{k})}e^{ikx_2}\right)$$

$$\left[\phi_{(t_1,\vec{x}_1)}\ ;\phi_{(t_2,\vec{x}_2)}\right]=\left[\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right)\ ,\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]$$

$$\left[\phi_{(t_1,\vec{x}_1)}\ ;\phi_{(t_2,\vec{x}_2)}\right]=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\left[\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]$$

$$\begin{aligned}\left[\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]&\\ =\left[\left(a_{(\vec{k})}e^{-ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}\right)\right]&+\left[\left(a_{(\vec{k})}e^{-ikx_1}\right),\left(a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]\\ +\left[\left(a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}\right)\right]&+\left[\left(a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]\end{aligned}$$

Como:

$$\left[a_{(\vec{k})},a_{(\vec{q})}\right]=\left[a^{\dagger}_{(\vec{k})},a^{\dagger}_{(\vec{q})}\right]=0$$

$$\left[a_{(\vec{k})},a^{\dagger}_{(\vec{q})}\right]=(2\pi)^3\delta^{(3)}_{(\vec{k}-\vec{q})}$$

Entonces:

$$\begin{aligned}\left[\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]&\\ =0+e^{-ikx_1}e^{ikx_2}\left[a_{(\vec{k})},a^{\dagger}_{(\vec{q})}\right]&+e^{ikx_1}e^{-ikx_2}\left[a^{\dagger}_{(\vec{k})},a_{(\vec{q})}\right]+0\end{aligned}$$

$$\left[\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]=e^{-ikx_1}e^{ikx_2}\left[a_{(\vec{k})},a^{\dagger}_{(\vec{q})}\right]-e^{ikx_1}e^{-ikx_2}\left[a_{(\vec{q})},a^{\dagger}_{(\vec{k})}\right]$$

$$\left[\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]=e^{ik(x_2-x_1)}\left[a_{(\vec{k})},a^{\dagger}_{(\vec{q})}\right]-e^{-ik(x_2-x_1)}\left[a_{(\vec{q})},a^{\dagger}_{(\vec{k})}\right]$$

$$\left[\left(a_{(\vec{k})}e^{-ikx_1}+a^{\dagger}_{(\vec{k})}e^{ikx_1}\right),\left(a_{(\vec{q})}e^{-ikx_2}+a^{\dagger}_{(\vec{q})}e^{ikx_2}\right)\right]=e^{ik(x_2-x_1)}(2\pi)^3\delta^{(3)}_{(\vec{k}-\vec{q})}-e^{-ik(x_2-x_1)}(2\pi)^3\delta^{(3)}_{(\vec{q}-\vec{k})}$$

Considerando que la función delta es par

$$\left[\phi_{(t_1,\vec{x}_1)}\,;\phi_{(t_2,\vec{x}_2)}\right]=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\left(e^{ik(x_2-x_1)}(2\pi)^3\delta^{(3)}_{(\vec{k}-\vec{q})}-e^{-ik(x_2-x_1)}(2\pi)^3\delta^{(3)}_{(\vec{k}-\vec{q})}\right)$$

$$\left[\phi_{(t_1,\vec{x}_1)}\,;\phi_{(t_2,\vec{x}_2)}\right]=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(e^{ik(x_2-x_1)}-e^{-ik(x_2-x_1)}\right)\int\frac{d^3q}{\sqrt{2\omega_q}}\delta^{(3)}_{(\vec{k}-\vec{q})}$$

$$\left[\phi_{(t_1,\vec{x}_1)}\,;\phi_{(t_2,\vec{x}_2)}\right]=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(e^{ik(x_2-x_1)}-e^{-ik(x_2-x_1)}\right)\frac{1}{\sqrt{2\omega_k}}$$

$$\boxed{\left[\boldsymbol{\phi}_{(t_1,\vec{x}_1)}\,;\boldsymbol{\phi}_{(t_2,\vec{x}_2)}\right]=\int\frac{\boldsymbol{d^3k}}{(\boldsymbol{2\pi})^3}\frac{\boldsymbol{1}}{\boldsymbol{2\omega_k}}\left(\boldsymbol{e^{ik(x_2-x_1)}-e^{-ik(x_2-x_1)}}\right)}$$

El mismo resultado que Javier adelantara en el curso, con $x_1 = 0$ y $x_2 = x$

EJERCICIO 3 (40:22)

Demostrar que:

$$\int \boldsymbol{d^3k \frac{1}{2\omega_k}} = \int \boldsymbol{d^4k\, \delta_{(\omega^2-|k|^2-m^2)}\theta_{(\omega)}}$$

Donde:

$$\boldsymbol{\theta_{(\omega)}} = \begin{cases} \boldsymbol{0}, & \boldsymbol{\omega < 0} \\ \boldsymbol{1}, & \boldsymbol{\omega \geq 0} \end{cases}$$

$$\int d^4k\, \delta_{(\omega^2-|k|^2-m^2)}\theta_{(\omega)} = \int d^3k \int d\omega\, \delta_{(\omega^2-|k|^2-m^2)}\theta_{(\omega)}$$

Según una de las fórmulas 36.11 del formulario de Crul, del curso de Javier de Mecánica Teórica (y ver ejemplo 1 del video, minuto 33:30)

$$\delta_{[f_{(x)}]}\Big|_u = \sum_i \frac{\delta_{(x-x_i)}}{|f'_{(x_i)}|}\Bigg|_u$$

Donde x_i son las raíces de f(x)

$$f_{(\omega)} = \omega^2 - |k|^2 - m^2$$

$$f'_{(\omega)} = 2\omega$$

Las raíces son:

$$\omega_1 = +\sqrt{|k|^2+m^2} = +\omega_k$$

$$\omega_2 = -\sqrt{|k|^2+m^2} = -\omega_k$$

$$\delta_{[f_{(\omega)}]} = \frac{\delta_{(\omega-\omega_1)}}{|f'_{(x_i)}|} + \frac{\delta_{(\omega-\omega_2)}}{|f'_{(x_i)}|} = \frac{\delta_{(\omega-(+\omega_k))}}{|2(+\omega_k)|} + \frac{\delta_{(\omega-(-\omega_k))}}{|2(-\omega_k)|}$$

$$\int d\omega\, \delta_{(\omega^2-|k|^2-m^2)}\theta_{(\omega)} = \int d\omega \left(\frac{\delta_{(\omega-(+\omega_k))}}{|2(+\omega_k)|} + \frac{\delta_{(\omega-(-\omega_k))}}{|2(-\omega_k)|}\right)\theta_{(\omega)}$$
$$= \int d\omega \frac{\delta_{(\omega-(+\omega_k))}}{|2(+\omega_k)|}\theta_{(\omega)} + \int d\omega \frac{\delta_{(\omega-(-\omega_k))}}{|2(-\omega_k)|}\theta_{(\omega)} = \frac{1}{2\omega_k}\left(\theta_{(+\omega_k)} + \theta_{(-\omega_k)}\right)$$
$$= \frac{1}{2\omega_k}(1+0)$$

$$\boxed{\int \boldsymbol{d^4k\, \delta_{(\omega^2-|k|^2-m^2)}\theta_{(\omega)}} = \int \boldsymbol{d^3k \frac{1}{2\omega_k}}}$$